

Bi-Hamiltonian aspects of the separability of the Neumann system

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Abstract

The Neumann system on the 2-dimensional sphere is used as a tool to convey some ideas on the bi-Hamiltonian point of view on separation of variables. It is shown that, from this standpoint, its separation coordinates and its integrals of motion can be found in a systematic way.

1 Introduction

Separation of variables for the Hamilton–Jacobi equation is a classical topic that is still very much investigated and has connections with several important research fields, such as stationary reductions of soliton equations (see [8] and references therein), algebraic completely integrable systems [1, 17], Riemannian geometry [3, 19, 20, 29], Bäcklund transformations and Baxter’s Q -operator [28, 21], and string theory [14].

The aim of this paper is to present, in the simple example of the Neumann system, the main ideas of a new approach [10, 22, 13] to separation of variables. This point of view is based on the geometry of bi-Hamiltonian manifolds, and has been successfully applied to the stationary reductions of the KdV hierarchy [11] and to Toda lattices [9].

The Neumann system is a well-known and very much studied mechanical system (see, e.g., [24, 2, 15, 26]), given by a (mass 1) particle moving on the (unit) sphere under the influence of a quadratic potential $V(x, y, z) = \frac{1}{2}(a_1x^2 + a_2y^2 + a_3z^2)$, where $a_1 < a_2 < a_3$ and (x, y, z) are Cartesian coordinates whose origin coincides with the center of the sphere. In 1859, Carl Neumann showed that the Hamilton–Jacobi equation of this system is additively separable in the so-called spheroconical (or elliptical spherical) coordinates (λ_1, λ_2) , given by

$$\frac{x^2}{\lambda - a_1} + \frac{y^2}{\lambda - a_2} + \frac{z^2}{\lambda - a_3} = \frac{(\lambda - \lambda_1)(\lambda - \lambda_2)}{(\lambda - a_1)(\lambda - a_2)(\lambda - a_3)},$$

with $a_1 < \lambda_1 < a_2 < \lambda_2 < a_3$. Once they have been introduced, one can check that they are separation coordinates, but the problem we want to address is to find out these coordinates in a systematic way. It is known that the separation variables of the Neumann system can be supplied by a Lax representation, but then the problem is to find such a representation for a given system, which is in general a quite difficult task. We will show that a careful study of the Neumann system will allow us to deduce in a natural way the spheroconical coordinates as separation variables.

The plan of the paper is the following. In Section 2 we will introduce the idea (due to A. Nijenhuis) of the geometrization of a coordinate system by means of a suitable tensor field L . This allows one to find a kind of compatibility condition between L and a given Hamiltonian H , ensuring that the coordinates induced by L separate the Hamilton–Jacobi equation associated with H . This condition is an intrinsic (i.e., coordinate-free) form of the classical Levi-Civita separability conditions. Before applying this results to the Neumann system in Section 4, we will comment on their meaning in the theory of bi-Hamiltonian manifolds in Section 3. Section 5 is devoted to two additional results on the Neumann system, that can be easily obtained from our standpoint (using some facts highlighted in [18]): A simple rule for the construction of the integrals of motion and the extension of the bi-Hamiltonian structure from the cotangent bundle T^*S^2 of the sphere to a 5-dimensional manifold, in order to give the Neumann system a bi-Hamiltonian formulation, which is missing in T^*S^2 . The final section contains some concluding remarks.

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2 The search for separation coordinates

Let \mathcal{Q} be an n -dimensional manifold, that can be thought of as the configuration space of a mechanical system (the sphere, for the Neumann system), and let H be a function on $T^*\mathcal{Q}$ (the Hamiltonian). In this section we will

present a method to look for the variables in which the Hamilton–Jacobi equation for H separates. This strategy will prove to be quite efficient in the case of the Neumann system.

The first important point, due to Nijenhuis [25], is the idea of the geometrization of a coordinate system. If $\{q^i\}$ are local coordinates on an open subset $U \subset \mathcal{Q}$, then the tensor field L of type $(1, 1)$ defined by

$$L \frac{\partial}{\partial q^i} = q^i \frac{\partial}{\partial q^i}$$

has vanishing Nijenhuis torsion, i.e.,

$$[LX, LY] - L[LX, Y] - L[X, LY] + L^2[X, Y] = 0$$

for every pair (X, Y) of vector fields on U . Viceversa, a tensor field L of type $(1, 1)$ whose torsion is zero induces local coordinates in a neighborhood of any point where the eigenvalues $(\lambda_1, \dots, \lambda_n)$ of L are distinct. In many cases these eigenvalues are functionally independent, so that

$$L \frac{\partial}{\partial \lambda_i} = \lambda_i \frac{\partial}{\partial \lambda_i} ,$$

i.e., they can be chosen as the coordinates associated with L . In the following we will suppose to be in this situation, that is, we will look for a tensor field L with vanishing torsion and functionally independent eigenvalues.

Once we have replaced the coordinate system with the geometric object L , it is quite natural to look for a “compatibility condition” between L and a function $H \in C^\infty(T^*\mathcal{Q})$ entailing that the coordinates given by L are separation variables for H . This condition can be obtained by lifting L from \mathcal{Q} to $T^*\mathcal{Q}$ by means of a procedure called “complete lifting” [30, 6]. It gives rise to a torsionless tensor field N of type $(1, 1)$ on $T^*\mathcal{Q}$, described in fibered coordinates (q^i, p_i) as

$$N \frac{\partial}{\partial q^i} = L_i^j \frac{\partial}{\partial q^j} + \left(\frac{\partial L_j^k}{\partial q^i} - \frac{\partial L_i^k}{\partial q^j} \right) p_k \frac{\partial}{\partial p_j} , \quad N \frac{\partial}{\partial p_i} = L_j^i \frac{\partial}{\partial p_j} .$$

Now that we have H and N on the same manifold $T^*\mathcal{Q}$, we can obtain the separability condition we are looking for as follows. First, we use H and N to construct the 2-form $\omega_H = d(N^*dH)$, where N^* is the adjoint of N . Then we consider the Hamiltonian vector field X_H and the vector fields NX_H , N^2X_H ,

... obtained by iteration, and we suppose that the distribution D_H spanned by $X_H, NX_H, \dots, N^{n-1}X_H$ is n -dimensional (i.e., that these vector fields are linearly independent at any point).

Theorem 1 *In the above-mentioned hypotheses, the coordinates associated with L are separation variables for H if and only if the 2-form ω_H annihilates the distribution D_H :*

$$\omega_H|_{D_H} = 0 . \quad (1)$$

The proof consists in writing the conditions

$$\omega_H(N^k X_H, N^l X_H) = 0 , \quad \text{for } k, l = 0, \dots, n-1, \quad (2)$$

in the canonical coordinates $(\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_n)$, where the λ_i are the eigenvalues of L and the μ_i are their conjugate momenta. Then, taking into account that N is diagonal in these coordinates, it is not difficult to realize that the equations (2) are equivalent to the classical Levi-Civita separability conditions ([8], p. 208).

Therefore, we have found the *separability condition* between L and H . It is simply the vanishing of a suitable 2-form on the distribution generated from X_H by means of the complete lift N of L . We stress that this condition is a concise and, above all, intrinsic form of the Levi-Civita equations. This means that one can check the separability of a given Hamiltonian H in the coordinates associated with a tensor field L *before* computing these coordinates. Moreover, one can impose condition (1) on L to search for separation coordinates for H . In Section 4 we will show how to exploit this fact in order to systematically deduce the separability of the Neumann system.

3 The bi-Hamiltonian meaning of the separability condition

Before applying the results of the previous section to the Neumann system, let us make some comments on their geometrical meaning.

1. Let \mathcal{Q} , L , and N be as in Section 2. Then the cotangent bundle of \mathcal{Q} is a bi-Hamiltonian manifold [18]. The first Poisson bracket is the canonical one,

$$\{F, G\} = \omega(X_F, X_G) ,$$

while the second one is given by

$$\{F, G\}' = \omega(NX_F, X_G) .$$

If P (resp. P') is the Poisson tensor of $\{\cdot, \cdot\}$ (resp. $\{\cdot, \cdot\}'$), this means that $P' = NP$.

2. Assume that H fulfills the separability conditions (1). Then the distribution D_H is integrable, so that there (locally) exist n independent functions H_1, \dots, H_n which are constant on the leaves of D_H . They are (local) first integrals for X_H . Notice however that the Hamiltonian vector field associated with the separable Hamiltonian H is not (in general, and in the particular example of the Neumann system) bi-Hamiltonian.

3. The distribution D_H is bi-Lagrangian, that is,

$$\{H_i, H_j\} = 0 , \quad \{H_i, H_j\}' = 0 \quad \text{for all } i, j.$$

As explained in [10, 13], this condition characterizes the (integrable) Hamiltonian systems that are separable in the coordinates induced by L . We remark in passing that bi-Lagrangian foliations play an important role in the study of special Kähler manifolds [16].

4. As we have already said in the previous section, the separation coordinates (λ_i, μ_i) are canonical for ω (this is obvious) and for N ,

$$N \frac{\partial}{\partial \lambda_i} = \lambda_i \frac{\partial}{\partial \lambda_i} , \quad N \frac{\partial}{\partial \mu_i} = \lambda_i \frac{\partial}{\partial \mu_i} .$$

They are often called Darboux-Nijenhuis coordinates (since N is sometimes called the Nijenhuis tensor) and have been used as separation coordinates in, e.g., [23, 4, 11, 12, 31].

4 The case of the Neumann system

In this section we will exploit the separability condition (1) to find the variables of separation for the Neumann system. To this aim, we will look for a torsionless tensor field L of type $(1, 1)$ on S^2 such that its complete lift N on T^*S^2 satisfies

$$\omega_H(X_H, NX_H) = 0 , \tag{3}$$

where $\omega_H = d(N^*dH)$ and H is the Neumann Hamiltonian. Due to the form of the constraint and of the potential, it is quite natural to perform the computations in the local coordinates $X = x^2$, $Y = y^2$, parametrizing every connected component of the subset obtained by removing from S^2 its intersections with the coordinate planes. In these coordinates the Hamiltonian of the Neumann system has the form:

$$H = 2 \left[X(1 - X)p_X^2 - 2XYp_Xp_Y + Y(1 - Y)p_Y^2 \right] + \frac{1}{2}(a_1 - a_3)X + \frac{1}{2}(a_2 - a_3)Y , \quad (4)$$

where (p_X, p_Y) are the conjugate momenta of (X, Y) . The unknown tensor field L can be written as

$$\begin{aligned} L^*(dX) &= AdX + BdY \\ L^*(dY) &= CdX + DdY , \end{aligned} \quad (5)$$

where (A, B, C, D) are functions of the coordinates (X, Y) that must satisfy two additional conditions. The first one is that the torsion of L has to vanish, that is,

$$\begin{aligned} B(C_Y - D_X) - BA_X - DA_Y + AA_Y + CB_Y &= 0 \\ C(B_X - A_Y) - AD_X - CD_Y + BC_X + DD_X &= 0 . \end{aligned}$$

The second one is the independency of the eigenvalues of L . Our aim is to find (A, B, C, D) in such a way to verify also the separability conditions (3). Thus we need to compute the complete lift N of L , which turns out to be given by

$$\begin{aligned} N^*dX &= AdX + BdY \\ N^*dY &= CdX + DdY \\ N^*dp_X &= -[p_X(B_X - A_Y) + p_Y(D_X - C_Y)]dY + Adp_X + Cdp_Y \\ N^*dp_Y &= [p_X(B_X - A_Y) + p_Y(D_X - C_Y)]dX + Bdp_X + Ddp_Y . \end{aligned} \quad (6)$$

Then (3) becomes a differential equation where the coordinates (X, Y) , the momenta (p_X, p_Y) , and the unknown functions (A, B, C, D) and their derivatives appear polynomially. This suggests to seek for a solution which also depends polynomially on the coordinates. Trying the simplest solution, one

finds

$$\begin{aligned}
A &= (a_3 - a_1)X + a_1 \\
B &= (a_3 - a_2)X \\
C &= (a_3 - a_1)Y \\
D &= (a_3 - a_2)Y + a_2 ,
\end{aligned} \tag{7}$$

which leads to the tensor field L defined by

$$\begin{aligned}
L^*(dX) &= a_1 dX + X d[(a_3 - a_1)X + (a_3 - a_2)Y] \\
L^*(dY) &= a_2 dY + Y d[(a_3 - a_1)X + (a_3 - a_2)Y] .
\end{aligned} \tag{8}$$

It can be checked that L can be extended to the whole sphere. Indeed, it is the conformal Killing tensor associated with the spheroconical coordinates [3]. They are the eigenvalues (λ_1, λ_2) of L , since

$$\begin{aligned}
\det(\lambda I - L) &= \lambda^2 - \lambda[(a_3 - a_1)X + (a_3 - a_2)Y + (a_1 + a_2)] \\
&\quad + a_2(a_3 - a_1)X + a_1(a_3 - a_2)Y + a_1 a_2 \\
&= \lambda^2(x^2 + y^2 + z^2) \\
&\quad - \lambda[(a_3 - a_1)x^2 + (a_3 - a_2)y^2 + (a_1 + a_2)(x^2 + y^2 + z^2)] \\
&\quad + a_2(a_3 - a_1)x^2 + a_1(a_3 - a_2)y^2 + a_1 a_2(x^2 + y^2 + z^2) \\
&= \lambda^2(x^2 + y^2 + z^2) - \lambda[(a_2 + a_3)x^2 + (a_1 + a_3)y^2 + (a_1 + a_2)z^2] \\
&\quad + [a_2 a_3 x^2 + a_1 a_3 y^2 + a_1 a_2 z^2] \\
&= (\lambda - a_2)(\lambda - a_3)x^2 + (\lambda - a_1)(\lambda - a_3)y^2 + (\lambda - a_1)(\lambda - a_2)z^2 \\
&= \left[\frac{x^2}{\lambda - a_1} + \frac{y^2}{\lambda - a_2} + \frac{z^2}{\lambda - a_3} \right] (\lambda - a_1)(\lambda - a_2)(\lambda - a_3) .
\end{aligned}$$

Hence,

$$\frac{(\lambda - \lambda_1)(\lambda - \lambda_2)}{(\lambda - a_1)(\lambda - a_2)(\lambda - a_3)} = \frac{x^2}{\lambda - a_1} + \frac{y^2}{\lambda - a_2} + \frac{z^2}{\lambda - a_3} ,$$

meaning that the eigenvalues of L are the spheroconical coordinates on S^2 . We have thus deduced the usual separation coordinates of the Neumann system simply using the separability condition (1) discussed in Section 2.

5 Integrals of motion for the Neumann system

The aim of this section is to show that in our setting one can easily describe other interesting features of the Neumann system, such as a simple

construction of the integrals of motion and the existence of a bi-Hamiltonian formulation in a suitable extension of the phase space. The crucial point is that the Hamiltonian H of the Neumann system and the complete lift N of the tensor field (8) satisfy the stronger condition

$$d(N^*dH) = dp_1 \wedge dH , \quad \text{where } p_1 = \text{tr } L = \lambda_1 + \lambda_2, \quad (9)$$

implying the separability condition (1). This fact has important consequences, that we are going to show in the general setting considered in Sections 2 and 3.

Let L be a torsionless tensor field of type $(1, 1)$, with functionally independent eigenvalues, on an n -dimensional manifold \mathcal{Q} , so that we can endow the symplectic manifold $(T^*\mathcal{Q}, \{\cdot, \cdot\})$ with the additional Poisson bracket

$$\{F, G\}' = \omega(NX_F, X_G) ,$$

using the complete lift N of L . Let

$$\det(\lambda I - L) = \lambda^n - (p_1\lambda^{n-1} + p_2\lambda^{n-2} + \cdots + p_n)$$

be the characteristic polynomial of L , and suppose that $H \in C^\infty(T^*\mathcal{Q})$ satisfies

$$d(N^*dH) = dp_1 \wedge dH . \quad (10)$$

Then one can easily show that $\omega_H = d(N^*dH)$ vanishes on the distribution D_H , so that the separability condition (1) is fulfilled and H is separable in the coordinates associated with L . We know from Section 3 that there are (local) integrals of motion for X_H . If (10) holds, they can be easily found by writing it in the form $d(N^*dH - p_1dH) = 0$, and choosing H_2 such that $dH_2 = N^*dH - p_1dH$. Then (as shown in [18] in the case where H is quadratic in the momenta) also the 1-form $N^*dH_2 - p_2dH$ is closed, and the process can be iterated, so that the integrals of motion for X_H can be determined from the recursion relations

$$\begin{aligned} dH_2 &= N^*dH - p_1dH \\ dH_3 &= N^*dH_2 - p_2dH \\ &\vdots \\ dH_n &= N^*dH_{n-1} - p_{n-1}dH . \end{aligned} \quad (11)$$

Moreover, we have that

$$0 = N^*dH_n - p_ndH . \quad (12)$$

For the Neumann system the global existence of the function H_2 is guaranteed by the fact that T^*S^2 is simply connected. One finds that, in local coordinates,

$$\begin{aligned} H_2 = & 2(a_1p_Y^2Y + a_2p_X^2X)(X + Y - 1) - 2a_3XY(p_Y - p_X)^2 \\ & - \frac{1}{2}a_2(a_1 - a_3)X - \frac{1}{2}a_1(a_2 - a_3)Y , \end{aligned}$$

which coincides with the constant of motion provided by the Lax matrix. Notice that the Hamiltonian vector field X_H is not bi-Hamiltonian on $T^*\mathcal{Q}$. Nevertheless, from (12) we can conclude that

$$X_H = \{\cdot, H\} = \frac{1}{p_n}\{\cdot, H_n\}' .$$

In the terminology of [5], this means that X_H is a quasi-bi-Hamiltonian vector field (see also [4, 23]). Next we will show that we can obtain a bi-Hamiltonian representation of X_H on the extended phase space $\mathcal{M} := T^*\mathcal{Q} \times \mathbb{R}$, following [18, 7]. The first step is the extension of the bi-Hamiltonian structure from $T^*\mathcal{Q}$ to \mathcal{M} . If $c \in \mathbb{R}$ is a coordinate in the “additional dimension” and $F \in C^\infty(T^*\mathcal{Q})$, then the new Poisson brackets are defined as

$$\begin{aligned} \{F, c\}_{\mathcal{M}} &= 0 \\ \{F, c\}'_{\mathcal{M}} &= \{F, H\} - c\{F, p_1\} . \end{aligned}$$

They endow \mathcal{M} with a bi-Hamiltonian structure. The second step is to notice that the recursion relations (11) on the H_i become the usual Lenard relations on the functions \hat{H}_i defined on \mathcal{M} as

$$\hat{H}_0 = c , \quad \hat{H}_i = H - cp_i \quad \text{for } i = 1, \dots, n.$$

Indeed, we have

$$\begin{aligned} \{\hat{H}_0, \cdot\}_{\mathcal{M}} &= 0 \\ \{\hat{H}_1, \cdot\}_{\mathcal{M}} &= \{\hat{H}_0, \cdot\}'_{\mathcal{M}} \\ &\vdots \\ \{\hat{H}_n, \cdot\}_{\mathcal{M}} &= \{\hat{H}_{n-1}, \cdot\}'_{\mathcal{M}} \\ 0 &= \{\hat{H}_n, \cdot\}'_{\mathcal{M}} . \end{aligned}$$

In the Neumann case, the restriction to $c = 0$ of the bi-Hamiltonian vector field $\{\hat{H}_1, \cdot\}_{\mathcal{M}} = \{\hat{H}_0, \cdot\}'_{\mathcal{M}}$ is the Neumann vector field. Thus we have shown that the Neumann system admits a bi-Hamiltonian formulation in an extended phase space, recovering in this way a result of [4].

6 Concluding remarks

1. The extension process of a chain of the form (11) into a Lenard chain is the opposite of the reduction technique presented in [11, 9], where a suitable quotienting produces a chain like (11) from a Lenard chain.
2. All the results presented here can be extended to the n -dimensional Neumann system. It would be interesting to compare our approach with the one based on the Lax representation.
3. The condition (9) appears also in [7], in the case where (\mathcal{Q}, g) is a Riemannian manifold and $H = \frac{1}{2}g^{ij}(q)p_i p_j + V(q)$. The authors show that (9) implies that L is a conformal Killing tensor of g with vanishing torsion. Notice however that in our approach to separability the Riemannian structure of \mathcal{Q} plays no role and the important objects are bi-Lagrangian foliations on bi-Hamiltonian manifolds.
4. The idea of extending the phase space in order to obtain a bi-Hamiltonian formulation of a given (Hamiltonian) vector field goes back, to the best of our knowledge, to [27], where it has been used also in the context of separation of variables.

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